THE APPLICATION OF ELECTROMAGNETIC THEORY TO ELECTROCARDIOLOGY

I. Derivation of the Integral Equations

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ABSTRACT One of the fundamental problems of theoretical electrocardiology is to determine the potential distribution on the surface of the torso due to the time-varying dipolar heart source. In this paper we present a rigorous derivation of the integro-differential equations for the potential, containing, for the first time, the effects of the time dependence of the source and the dielectric properties of the medium. These equations provide a general and rigorous basis from which to attack the problem numerically on a computer and permit the use of a detailed model of the thorax as a multiple region volume of different dielectric and conducting properties.

I. INTRODUCTION

One of the fundamental problems of theoretical electrocardiology is to determine the potential distribution over the surface of the torso produced by an internal time-varying source (dipolar, quadrupolar, etc.). Historically, this problem has been attacked analytically by making simplifying assumptions about both the electrical and geometrical properties of the thorax (see, for example, the excellent bibliography in reference 11). Approaching the problem in this way, such assumptions as an infinite homogeneous medium or a finite homogeneous medium of spherical or prolate spheroidal shape have been made.

More recently, it has been recognized that the computer may be a useful tool in attacking the problem, since more complex geometries and electrical properties may be assumed. These may be closer representations of the human physiology; that is, time-dependent sources embedded within an inhomogeneous conducting and dielectric medium of irregular configuration. Gelernter and Swihart (4) used impressive physical intuition to derive a system of equations (the G-S equations) for the charge produced on each element of a surface mosaic, and from this to derive the potential distribution; however, time dependence and dielectric effects were ignored.

In this paper we give a rigorous derivation of the integro-differential equations which relate the sources to the potentials produced on the surfaces of an inhomogeneous medium of arbitrary shape. In the derivation we include the effect of the time dependence of the cardiac electrical generator (and of the resulting charges and potentials) and the effect of the dielectric properties of the torso. Other authors (2, 4–6) have heuristically ignored time dependence at the outset, implicitly assuming that a small term in a partial differential expression only produces small perturbations upon solutions of the resulting equations. This is not necessarily true, and it is preferable (if possible) to retain the time-dependent terms and examine their importance a posteriori. Even if we make the customary assumption that time-dependent effects are negligible, it is by no means obvious that the dielectric effects (induced polarization) can be ignored.

There is no compelling reason to make a priori assumptions about the time-dependent terms arising from the time variation of the source. These terms only slightly complicate the expressions, and they can easily be carried through the derivation. It is found in this paper that the time effects and dielectric effects enter the integro-differential equations in such an interrelated manner, that if time-dependent terms can be ignored, then essentially so may the dielectric effects. However, if these terms cannot be ignored, the dielectric effects enter the resulting time-dependent equations in a somewhat complex manner.

Ideally, we should adopt the above philosophy when considering the term $\partial \mathbf{B}/\partial t$ in Maxwell's equations, i.e., we should carry it during the development and assess its importance by examining the final result. Unfortunately, including this term makes the problem intractable, and we have no alternative but to set it to 0 at the outset. In Appendix I, we give some justification for this.

In the context of human electrophysiology, there is some experimental evidence to indicate that all of the time-dependent terms are probably negligible. However, this has not been shown rigorously. Mathematically it is a complex property of the integro-differential operators we derive. If time *can* be ignored, we have established the results mentioned above, that the dielectric properties of the media vanish from the equations. However, if the time-dependent effects are *not* negligible, particularly in the context of the time-dependent (as opposed to a constant) source, then we provide a tool (equations 35 or 56) for further study of the problem.

The layout of the paper is as follows. In section II we state the time-dependent boundary conditions, using familiar results to establish the notation. In section III we derive the integro-differential equation for the surface densities of charge on the interfaces between n regions. From these charge densities the resulting potentials are easily obtained. Integro-differential equations which give the potentials directly are derived in section IV.

The G-S equations are a discrete analogue of the steady-state limit of our equations (section III) with the dielectric effects ignored. In reference 4 and other papers (5, 6), Gelerntner and Swihart also discussed the numerical solution of their equations for several geometries, some simple enough to be checked against known analytical results. It can be shown (1, 9) that the numerical techniques suggested in reference 4 are adequate only when simple geometries, such as spheres, are being considered. The geometries involved in the human torso are sufficiently complex to demand the use of more sophisticated numerical techniques.

Barr et al. (2) have used Green's theorem to derive an integral equation which gives the potential distribution directly. They obtain equations for one- and two-region geometries without time-dependent effects. These equations, which are again subtle to treat numerically (1, 9), are special cases of the equations derived in section IV.

Our treatment of integrals with singular integrands is somewhat cavalier, since we have attempted not to obscure the main points of this paper by a high degree of mathematical precision. However, all the steps that have been taken in this slightly imprecise manner can be rigorously justified (8).

II. BASIC ELECTROMAGNETIC THEORY

The purpose of this section is to derive from Maxwell's equations (7, 10) an equation for the potential produced by a dipole source embedded in a dielectric (equation 17). Boundary conditions are desired for interfaces where there are discontinuities in the dielectric or conducting properties of the medium (equations 18 and 19). We shall thus arrive at a suitable starting point from which, in sections III and IV, we derive the various integro-differential equations.

For simplicity the medium will be assumed to be linear, isotropic, and homogeneous, although ideally this is probably not a good assumption. A muscle fiber, for example, would not be expected to have isotropic properties. However, although the necessary extension of the theory to include anisotropic properties is straightforward, the distribution of the anisotropies appears to be so fragmentary and the knowledge thereof so incomplete that it would be impossible to translate the theory into practice. Thus, in the present treatment, regions of the body such as the lungs or cardiac ventricles will be taken to have homogeneous and isotropic properties.

A. *Basic Equations*

In Appendix I, some order-of-magnitude estimates establish that $\nabla \times \mathbf{E} = 0$, and so $\mathbf{E}$ may be obtained from a scalar potential function

$$\mathbf{E} = -\nabla\phi. \tag{1}$$

The potential due to a continuous volume distribution of charge ρ throughout the volume V and a surface distribution of charge ω on various surfaces S is

$$\begin{aligned}\phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\int \frac{dq(\mathbf{r}')}{|r - r'|} \\ &= \frac{1}{4\pi\epsilon_0}\left\{\int_V \frac{\rho(\mathbf{r}')\,dV'}{|\mathbf{r} - \mathbf{r}'|} + \int_S \frac{\omega(\mathbf{r}')\,dS'}{|\mathbf{r} - \mathbf{r}'|}\right\}.\end{aligned} \tag{2}$$

In particular, for a two-point charge distribution (located at $\mathbf{r}'$ and $\mathbf{r}_1'$) in a vacuum

$$\rho(\mathbf{r}) = q\{\delta(\mathbf{r} - \mathbf{r}') - \delta(\mathbf{r} - \mathbf{r}_1')\} \tag{3}$$

where δ is the Dirac delta function. The electric dipole moment is defined by

$$\mathbf{M} = q(\mathbf{r}' - \mathbf{r}_1') \tag{4}$$

and the potential at a field point $\mathbf{r}$ in the limit of vanishing separation

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{M} \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{5}$$

A volume distribution of polarization with electric dipole moment per unit volume $\mathbf{m}(\mathbf{r}')$ produces a potential

$$\begin{aligned}\phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int_V \frac{\mathbf{m}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, dV' \\ &= \frac{1}{4\pi\epsilon_0} \int_V \mathbf{m} \cdot \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \, dV'.\end{aligned} \tag{6 a}$$

The integrals in equation 6 *a* have singular integrands when the field point $\mathbf{r}$ coincides with the source point $\mathbf{r}'$. The prescription for handling these singularities is developed in Appendix III. Using the Divergence Theorem, we may alternatively write equation 6 *a* as

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left\{ \int \frac{\mathbf{m} \cdot \mathbf{dS}}{|\mathbf{r} - \mathbf{r}'|} + \int \frac{-\nabla \cdot \mathbf{m}}{|\mathbf{r} - \mathbf{r}'|} \, dV' \right\}. \tag{6 b}$$

Equations 6 *a* and 6 *b* are exactly equivalent expressions for the potential due to a specified distribution of polarization.

B. *Dielectric Media*

The above equations for the field produced by electric dipoles in a vacuum are independent of the source of the dipoles. In a *dielectric* medium, however, the polarization is *induced* by the electric field. Thus the volume density of polarization is a function of the potential ϕ, and both sides of equation 6 contain the unknown potential.

The susceptibility tensor χ relates the *induced* dipole moment per unit volume $\mathbf{P}$ to $\mathbf{E}$

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E} = -\epsilon_0 \chi \nabla \phi. \tag{7}$$

For linear, isotropic, and piecewise homogeneous media, χ reduces to a piecewise constant scalar. Defining **D** by

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \tag{8}$$

we have

$$\mathbf{D} = \epsilon_0(1 + \chi)\mathbf{E} = K\epsilon_0\mathbf{E} = \epsilon\mathbf{E} \tag{9}$$

where $K = 1 + \chi$ is the dielectric constant (for a particular region) and

$$\epsilon = K\epsilon_0 \tag{10}$$

is the permittivity of the dielectric with ϵ_0 the permittivity of the vacuum. (Note: for air, K is very close to 1; for water and presumably body tissue, K is about 80. It certainly seems, therefore, that any theoretical treatment of electrophysiology should include a discussion of dielectric effects; this is one of the objectives of the present paper.) Then Maxwell's equation

$$\nabla \cdot \mathbf{D} = \rho \tag{11}$$

becomes

$$\nabla \cdot \mathbf{E} = \rho/\epsilon \tag{12}$$

and

$$\nabla^2\phi = -\rho/\epsilon. \tag{13}$$

Now consider a point charge situated at the origin and embedded in dielectric material. In this case

$$\rho(\mathbf{r}') = q\delta(\mathbf{r}'), \tag{14}$$

and from equations 7, 10, and 12

$$\nabla \cdot \mathbf{P} = (\chi/K)q\delta(\mathbf{r}'). \tag{15}$$

The potential at **r** due to a point charge and due to the induced polarization is obtained using equations 2, 6 *b*, and 15 as

$$\begin{aligned}\phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\left(1 - \frac{\chi}{K}\right)\frac{q}{|\mathbf{r}|} + \frac{1}{4\pi\epsilon_0}\int_S \frac{\mathbf{P}\cdot d\mathbf{S}'}{|\mathbf{r} - \mathbf{r}'|} \\ &= \frac{1}{4\pi\epsilon}\frac{q}{|\mathbf{r}|} + \frac{1}{4\pi\epsilon_0}\int_S \frac{\chi\epsilon_0\,(-\partial\phi/\partial n)\,dS'}{|\mathbf{r} - \mathbf{r}'|}\end{aligned} \tag{16}$$

where $-\ \partial\phi/\partial n$ is the component of **E** in the direction of the outward-drawn normal to the surface. It will often be convenient to write $E_n = -\frac{\partial \phi}{\partial n}$. Note that the unknown ϕ appears on both sides of equation 16.

We now consider both charges q and noninduced dipoles $\mathbf{m}$ embedded in the dielectric. In the general case there may be several dielectric regions V_i ($i = 1, \cdots I$) each bounded by several surfaces S_{ij} ($j = 1 \cdots J(i)$). The result is

$$\phi(\mathbf{r}) = \sum_{i=1} \left\{ \frac{1}{4\pi\epsilon_i} \int_{V_i} \frac{dq_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{4\pi\epsilon_i} \int_{V_i} \frac{(\mathbf{r} - \mathbf{r}') \cdot d\mathbf{m}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} + \sum_{j=1}^{J(i)} \frac{1}{4\pi\epsilon_0} \int_{S_{ij}} \frac{\chi_i \epsilon_0 \left(-\frac{\partial \phi}{\partial n} \right)_{ij}}{|\mathbf{r} - \mathbf{r}'|} dS_{ij} \right\} \tag{17}$$

Here $\left(-\frac{\partial \phi}{\partial n} \right)_{ij}$ is the component of the electric field in region (i) in the direction of the outward-drawn normal to the surface (ij).

C. *Boundary Conditions for Dielectrics with Finite Conductivity*

Previously we have assumed the media to be nonconductors. Let us consider now dielectric media with nonzero conductivity σ. When an electric field is applied, two effects occur: polarization is induced in the media as discussed above, but in addition there will be some migration of free charges because of the finite conductivity. It is easily shown (Appendix II) that the volume charge density within a conductor is always 0.

At the boundary between two such regions Gauss's law and charge conservation together with Ohm's law provide the necessary boundary conditions relating the electric field strengths on the two sides of the boundary. Consider regions (i) and (j) separated by the surface (ij). Call the field in region (i) at the surface (ij) in the direction of the outward-drawn normal to the surface, i.e. from (i) to (j), E_{ij} (and similarly E_{ji}). The properties of the region (i) are labeled ϵ_i, σ_i, and the surface charge density on the surface (ij) is called ω_{ij}. In Fig. 1 a "pill box" is indicated which has been flattened to the limit where the curved surface approaches zero area. Applying equation 11 and using the divergence theorem, we obtain in the limit

$$\epsilon_i E_{ij} + \epsilon_j E_{ji} = -\omega_{ij}. \tag{18}$$

Similarly applying the charge continuity equation and the divergence theorem to the "pill box"

$$\sigma_i E_{ij} + \sigma_j E_{ji} = \frac{\partial \omega_{ij}}{\partial t}, \tag{19}$$

where the conductivity σ is defined through Ohm's law

$$\mathbf{j} = \sigma \mathbf{E}. \tag{20}$$

Solving equations 18 and 19 for the normal components of **E** on the two sides of the boundary, we have

$$E_{ij} = \left(\sigma_j \omega_{ij} + \epsilon_j \frac{\partial \omega_{ij}}{\partial t}\right) \Big/ (\sigma_i \epsilon_j - \sigma_j \epsilon_i) \tag{21}$$

and similarly for E_{ji}.

It is tempting to drop the time-dependent part of the boundary condition, since the ratio of the first to the second term in equation 21 is $\sigma/2\pi f\epsilon$ where f is the frequency of a typical Fourier component. For electrocardiological frequencies this ratio is probably much greater than unity (see Appendix II). However at this point

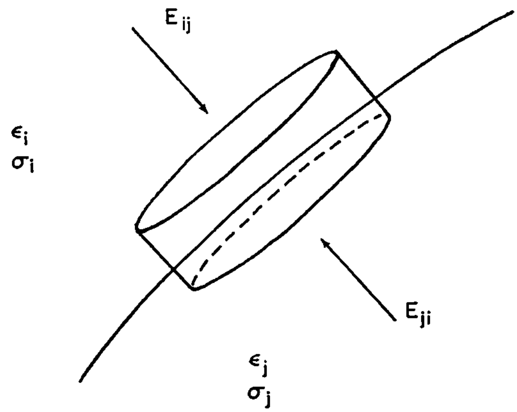

FIGURE 1 At the surface between region i and j Gauss's theorem is applied to a "pill box."

we cannot be sure that a small difference in the boundary conditions produces only small differences in the equations obtained later. In the interests of generality (and at little cost in complication) we retain the time-dependent term.

Equation 17 together with the boundary conditions, equation 21, provide the basis for the following derivations.

III. THE INTEGRAL EQUATION FOR CHARGE

In this section an integro-differential equation will be derived for the surface charge distribution due to a dipole source in conducting dielectric medium. The numerical solution of this equation is outlined in the forthcoming paper (1). For simplicity the case of a single region (numbered 2) surrounded by air (numbered 1) will be treated first. This would be applicable if the torso were modeled as a single homogeneous region. The multiregion case, applicable to an inhomogeneous torso model, will be discussed in section III B.

A. *Homogeneous Conducting Dielectric Surrounded by Air*

Consider (see Fig. 2) noninduced polarization $\mathbf{m}$ situated in region 2 (permittivity ϵ_2, finite conductivity σ_2) surrounded by air in region 1 ($\epsilon_1 = \epsilon_0$, $\sigma_1 = 0$). We adhere to the notation introduced in section II C. The potential is, from equation 17,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_2}\int_{V_2}\frac{(\mathbf{r}-\mathbf{r}')\cdot d\mathbf{m}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} + \frac{1}{4\pi\epsilon_0}\int_{S_{12}}\frac{\omega_{12}(\mathbf{r}')dS_{12}}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi\epsilon_0}\int_{S_{12}}\frac{\chi_2\,\epsilon_0\,E_{21}}{|\mathbf{r}-\mathbf{r}'|}\,dS_{12} \quad (22)$$

subject to the boundary conditions, equations 18 and 19. Consider a point $\mathbf{r}$ just

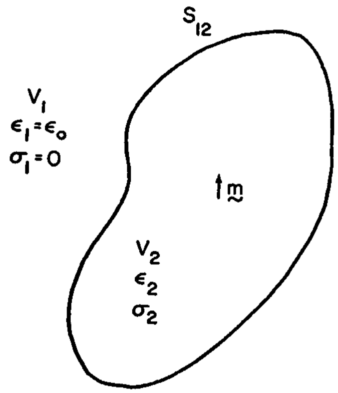

FIGURE 2 Noninduced polarization $\mathbf{m}$ is situated in region *2* (permittivity ϵ_2, finite conductivity σ_2), surrounded by air in region *1* ($\epsilon_1 = \epsilon_0$ $\sigma_1 = 0$).

outside the surface S_{12}. (Let us write $|\mathbf{r}-\mathbf{r}'| = r$; we shall denote a normal derivative in V_1 at the surface S_{12} in the direction from 1 to 2 by the operator ∂_{12}. Note that $\partial_{12} = -\partial_{21}$ only when operating on functions which have a continuous derivative across the boundary S_{12}.) Then from equation 22 with a boundary condition, equation 21, for E_{21} we get

$$E_{12} = -\partial_{12}\,\phi$$

$$= -\partial_{12}\,\phi_m - \frac{1}{4\pi\epsilon_0}\,\partial_{12}\int_{S_{12}}\frac{\omega_{12}}{r}\,dS_{12} - \frac{1}{4\pi\epsilon_0}\,\partial_{12}\int_{S_{12}}\chi_2\,\epsilon_0\,\frac{1}{\sigma_2}\,\frac{\partial\omega_{12}}{\partial t}\,\frac{dS_{12}}{r} \quad (23)$$

where

$$\phi_m = \frac{1}{4\pi\epsilon_2}\int_{V_2}\frac{(\mathbf{r}-\mathbf{r}')\cdot d\mathbf{m}}{|\mathbf{r}-\mathbf{r}'|^3} \quad (24)$$

is the potential which would be produced by the sources in an infinite medium.

In the limit that $\mathbf{r}$ is on the surface S_{12}, which is the case of interest, the integrands in equation 23 have a singularity at $\mathbf{r}' = \mathbf{r}$. It can be shown, as is done in Appendix III, that

$$\partial_{21} \int_{S_{12}} \psi(\mathbf{r}') \frac{1}{r} \, dS_{12} = \int_{S_{12}} \psi(\mathbf{r}') \, \partial_{21}\left(\frac{1}{r}\right) dS_{12} + 2\pi\psi(\mathbf{r}) \tag{25}$$

for suitably smooth functions ψ for which the proper value of the integral (implied on the right) exists. Now to satisfy the boundary condition, equation 21,

$$E_{12} = -\frac{1}{\epsilon_0}\left\{\omega_{12} + \frac{\epsilon_2}{\sigma_2}\frac{\partial\omega_{12}}{\partial t}\right\}. \tag{26}$$

Equations 23 and 26 provide an integro-differential equation for the surface charge ω_{12} :

$$\frac{\omega_{12}(\mathbf{r})}{2\epsilon_0} + \frac{(2+\chi_2)}{2\sigma_2}\frac{\partial\omega_{12}(\mathbf{r})}{\partial t} = \partial_{12}\,\phi_m(\mathbf{r}) + \int_{S_{12}} K_{21}(\mathbf{r},\mathbf{r}')\,\frac{\omega_{12}(\mathbf{r}')}{2\epsilon_0}\,dS'_{12}$$

$$+ \int_{S_{12}} K_{21}(\mathbf{r},\mathbf{r}')\,\frac{\chi_2}{2\sigma_2}\frac{\partial}{\partial t}\,\omega_{12}(\mathbf{r}')\,dS'_{12} \tag{27}$$

with

$$K_{21}(\mathbf{r},\mathbf{r}') = -\frac{1}{2\pi}\,\partial_{21}\frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{28}$$

for a field point $\mathbf{r}$ on the surface S_{12}. In the steady-state limit $\frac{\partial\omega_{12}}{\partial t} = 0$ and equation 27 becomes an integral equation for the unknown surface charge density on S_{12}. If this equation can be solved for a given source potential ϕ_m, then the potential at any point $\mathbf{r}$ can be evaluated using equation 22 with $E_{21} = 0$. It is shown in Appendix II that the $\mathbf{r}' = \mathbf{r}$ singularity in the integrand causes no difficulty in the evaluation of $\phi(\mathbf{r})$ for $\mathbf{r}$ belonging to the surface.

Note that in the steady-state limit the only reference to the dielectric properties of region 2 lies in the source term ϕ_m. Provided only relative source strengths are of interest, the dielectric constant of region 1 need not be known in the steady-state case but only affects the transient behavior.

B. *Several Regions with Different Properties*

In the preceding section, region 2 was surrounded by region 1 which was taken to be air. We now generalize this result to the case where we have an arbitrary number of regions (i, j) separated by surfaces (ij). The reader might find it instructive to consider only one additional region, called 3, contained within region 2 (see Fig. 3). For example, region 3 might be the blood mass in the left ventricle. The potential has contributions from the sources, the charges on all surfaces and the normal components of induced polarization on all surfaces:

$$\phi(\mathbf{r}) = \phi_m(\mathbf{r}) + \frac{1}{4\pi\epsilon_0}\sum_{(ij)}\left\{\int_{S_{ij}} \omega_{ij}(\mathbf{r}')\,\frac{dS'_{ij}}{r} + \int_{S_{ij}} \epsilon_0(\chi_i E_{ij} + \chi_j E_{ji})\,\frac{dS'_{ij}}{r}\right\} \quad (29)$$

where again we are using the notation of section II C to label regions, surfaces, and normal components of the electric field at the surfaces. For example, E_{23} is the field in region 2 at the surface 23 in the direction of the normal to the surface 23 drawn from region 2 to region 3. The summation over $({}_{ij})$ in equation 29 includes just the two terms 12 and 23 in the simple three-region case. Now insert the boundary condi-

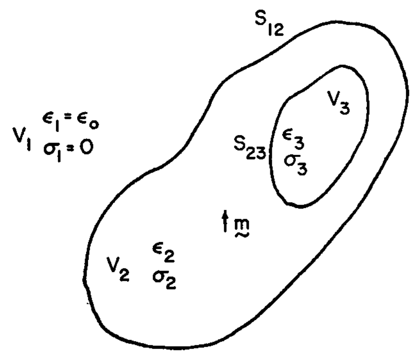

FIGURE 3 An additional region, labeled *3*, is added to the situation shown in Fig. 2.

tion, equation 21, for the E_{ij}

$$\phi(\mathbf{r}) = \phi_m(\mathbf{r}) + \frac{1}{4\pi\epsilon_0}\sum_{ij}\int_{S_{ij}} \frac{(\sigma_i - \sigma_j)\omega_{ij}(\mathbf{r}') + (\epsilon_i - \epsilon_j)\,\dfrac{\partial\omega_{ij}(\mathbf{r}')}{\partial t}}{(\sigma_i\chi_j - \sigma_j\chi_i)}\,\frac{dS'_{ij}}{r}. \quad (30)$$

We obtain an integral equation for the surface charges by forming E_{ij} from equation 30

$$E_{ij} = -\partial_{ij}\phi \quad (31)$$

and once again imposing the boundary condition of equation 21. If we treat the singularities in the integrands as in Appendix II, we obtain for $\mathbf{r}$ on S_{ij}

$$\frac{1}{2}\,\frac{(\sigma_i + \sigma_j)\omega_{ij}(\mathbf{r}) + (\epsilon_i + \epsilon_j)\,\dfrac{\partial\omega_{ij}(\mathbf{r})}{\partial t}}{(\sigma_i\epsilon_j - \sigma_j\epsilon_i)} = -\partial_{ij}\phi_m(\mathbf{r})$$

$$+\frac{1}{2}\sum_{(k\iota)}\int_{S_{k\iota}} \frac{(\sigma_k - \sigma_\iota)\omega_{k\iota}(\mathbf{r}') + (\epsilon_k - \epsilon_\iota)\,\dfrac{\partial\omega_{k\iota}(\mathbf{r}')}{\partial t}}{(\sigma_k\epsilon_\iota - \sigma_\iota\epsilon_k)}\,K_{ij}(\mathbf{r},\mathbf{r}')\,dS'_{k\iota}. \quad (32)$$

When forming the boundary condition (equation 21), it seemed clear that the time-dependent term should be negligible. However we see that the time-dependent terms appear in the equation in a more complicated way than in the boundary condition. On the left of equation 32 the relative size of the time-dependent term depends on $(\epsilon_i + \epsilon_j)/(\sigma_i + \sigma_j)$ which is certainly small. However on the right, relevant quantity is $(\epsilon_k - \epsilon_\iota)/(\sigma_k - \sigma_\iota)$, and it is less clear that the time-dependent term can be neglected.

Equation 32 reduces to equation 27 in the case where we have only two regions and one boundary surface, and we set $\sigma_1 = 0$, $\epsilon_1 = \epsilon_0$. In the three-region case we have a set of coupled equations for ω_{12} and ω_{23}. It is clear that equation 32 applies to the case of an arbitrary number of surfaces also, provided that we understand ϕ_m to be a generalized source potential (see equation 24)

$$\phi_m(\mathbf{r}) = \sum_i \frac{1}{4\pi\epsilon_i} \int_{V_i} \frac{(\mathbf{r} - \mathbf{r}')\cdot d\mathbf{m}_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{33}$$

and include all surfaces in the set (ij).

The equations 30 and 32 are simplified by the transformation

$$W_{ij}(\mathbf{r}) = \frac{(\sigma_i - \sigma_j)}{(\sigma_i\epsilon_j - \sigma_j\epsilon_i)} \frac{\omega_{ij}(\mathbf{r})}{2}. \tag{34}$$

Physically this corresponds to working in terms of "total" charge rather than the "free" charge. Then

$$\begin{aligned} \frac{\sigma_i + \sigma_j}{\sigma_i - \sigma_j} W_{ij}(\mathbf{r}) + \frac{\epsilon_i + \epsilon_j}{\sigma_i - \sigma_j} \frac{\partial}{\partial t} W_{ij}(\mathbf{r}) &= -\partial_{ij}\phi_m(\mathbf{r}) \\ &+ \sum_{(k\iota)} \int_{S_{k\iota}} K_{ij}(\mathbf{r}, \mathbf{r}') \left\{ W_{k\iota}(\mathbf{r}') + \frac{\epsilon_k - \epsilon_\iota}{\sigma_k - \sigma_\iota} \frac{\partial}{\partial t} W_{k\iota}(\mathbf{r}') \right\} dS'_{k\iota}. \end{aligned} \tag{35}$$

Equation 35 provides a basis for solving the general time-dependent problem with any number of regions, arbitrary in shape. Each region is assumed internally homogeneous, linear, and isotropic in its conducting and dielectric properties. Thus lungs and other organs may be represented.

The importance of the time-dependent terms is difficult to assess. Although these terms in the equation are probably small, it does not necessarily follow that the time-dependent solutions rapidly reach a steady state. This point is being studied further.

For static problems the system of equations with time dependence omitted is the exact, continuous form of the discrete equations obtained by Gelernter and Swihart (4). The pitfalls involved in replacing the continuous formulation by a discrete approximation appear, however, to have been ignored by these authors. The difficulties are discussed in the following paper (1). Note that if time dependence is omitted, the dielectric properties appear only in the source term.

The next section may provide a more efficient way to solve the problem. Instead of solving first for the surface charge densities and then obtaining the potential, we construct the integral equations directly for the potential. These equations, obtained in a limited form by Barr et al. (2), are also subtle to treat numerically, as pointed out in the following paper (1).

IV. THE INTEGRAL EQUATION FOR POTENTIAL

The integral equation for potential is obtained by applying Green's theorem successively to the various regions in the model torso. As in section III, the equation will be obtained first for a homogeneous torso surrounded by air and later for the general case. However, before deriving the equations we digress in order to discuss Green's theorem for a field point on the surface considered. The forms of the theorem are such that it is not obvious that the resulting potentials are, as they must be, continuous across the boundary, so it will be explicitly demonstrated that the potentials have this essential property.

A. *Green's Theorem for a Field Point on the Surface*

Apply the divergence theorem to the difference $\phi\nabla\psi - \psi\nabla\phi$ and obtain Green's theorem in the form

$$\int_V (\phi\nabla'^2\psi - \psi\nabla'^2\phi)\, dV' = \int_S (\phi\nabla'\psi - \psi\nabla'\phi)\cdot d\mathbf{S}'. \tag{36}$$

The operator ∇' acts on the coordinates $\mathbf{r}'$ of the source point. Consider $\psi = 1/|\mathbf{r} - \mathbf{r}'|$, which is written concisely as $1/r$. Then

$$\int_V \phi(\mathbf{r}')\nabla'^2\frac{1}{r}\, dV' = \int_V \frac{1}{r}\nabla'^2\phi(\mathbf{r}')\, dV + \int_S \left[\phi(\mathbf{r}')\nabla'\frac{1}{r} - \frac{1}{r}\nabla'\phi(\mathbf{r}')\right]\cdot d\mathbf{S}'. \tag{37}$$

Consider first the left-hand side of equation 37. Since

$$\nabla'^2\left(\frac{1}{r}\right) = -4\pi\delta(\mathbf{r} - \mathbf{r}'), \tag{38}$$

then

$$\int_V \phi(\mathbf{r}')\nabla'^2\left(\frac{1}{r}\right) dV' = \begin{cases} -4\pi\phi(\mathbf{r}) \\ 0 \end{cases} \tag{39}$$

depending on whether $\mathbf{r}$ is inside or outside V, respectively. The case of interest is when $\mathbf{r}$ is on the bounding surface S ($\mathbf{r}\epsilon S$) in which case equation (39) is ambiguous and, furthermore, singularities appear in the surface integral on the right-hand side of equation 37. The prescription developed in Appendix III leads to the equation for the potential on the surface

$$-2\pi\phi(\mathbf{r}\epsilon S) = \int_V \frac{1}{r}\nabla'^2\phi(\mathbf{r}')\,dV' + \int_S \left[\phi(\mathbf{r}')\nabla'\frac{1}{r} - \frac{1}{r}\nabla'\phi(\mathbf{r}')\right]\cdot d\mathbf{S}' \quad (40)$$

(see reference 3 for a more rigorous derivation).

Using equation 39, we obtain the potential at a point **r** within the volume V ($\mathbf{r}\epsilon V$) to be

$$-4\pi\phi(\mathbf{r}\epsilon V) = \int_V \frac{1}{r}\nabla'^2\phi(\mathbf{r}')\,dV' + \int_S \left[\phi(\mathbf{r}')\nabla'\frac{1}{r} - \frac{1}{r}\nabla'\phi(\mathbf{r}')\right]\cdot d\mathbf{S}'. \quad (41)$$

At first sight this appears inconsistent with equation 40, since **r** can be made arbitrarily close to S without affecting equation 41. Equations 40 and 41 might then seem to give different potentials for points arbitrarily close together. However, the second integral in equation 40 is the proper value, that is, it is taken over the diminished surface excluding the singularity. The integral in equation 41 is over the full surface S. This contributes a difference of just $-2\pi\phi(\mathbf{r})$ on the right-hand side and the equations are in fact consistent.

Note that the surface S which appears in Green's theorem are hypothetical and need not correspond to a physical surface. In particular the Green's theorem surface may be chosen just inside the physical surface if this is convenient.

Finally it may be noted that the potential given by equation 40 is continuous as the Green's theorem surface passes through a physical surface. For example, for a conducting region surrounded by a nonconducting region, there will be a layer of charge on the physical surface

$$\omega = -\epsilon_0(\nabla\phi)_n, \quad (42)$$

and inside the physical surface

$$(\nabla\phi)_n = 0. \quad (43)$$

If S is taken just inside the physical surface, equation (40) becomes [neglecting sources within the physical volume and using equation (21)]

$$-2\pi\phi(\mathbf{r}) = 0 + \int_S \left[0 + \phi\nabla'\left(\frac{1}{r}\right)\right]\cdot d\mathbf{S}'. \quad (44)$$

If S is taken just outside the physical surface, the charges on the physical surface are now within V and contribute in the first integral in equation 40

$$-2\pi\phi(\mathbf{r}) = \int_S \frac{-\omega(\mathbf{r}')}{\epsilon_0}\frac{1}{r}\,dS' + \int_S \left[-\frac{1}{r}\nabla'\phi + \phi\nabla'\frac{1}{r}\right]\cdot d\mathbf{S}' \quad (45)$$

which, using equation 42, is the same as equation 44.

Summing up this digression, we have presented Green's theorem for a point on the Green's theorem surface S, and we have verified that the resulting integral equations are consistent with continuity of the potential across the Green's theorem surface and also across a physical boundary with surface charge ω.

B. *Single Homogeneous Conducting Region Surrounded by Air*

Choose the Green's theorem surface S just inside the physical boundary. In the time-independent limit the normal derivative of the potential on S is 0 according to equation 21. Equation 40 becomes an integral equation with Neumann boundary conditions, and the potential is uniquely determined, apart from an unimportant additive constant. If the sources within the Green's theorem volume consist of polarization $\mathbf{m}(\mathbf{r}')$, then equation 40 becomes

$$\phi(\mathbf{r}) = 2\phi_m(\mathbf{r}) + \int_{S'} K'(\mathbf{r}, \mathbf{r}')\phi(\mathbf{r}')\, dS' \tag{46}$$

where

$$K'(\mathbf{r}, \mathbf{r}') = -\frac{1}{2\pi}\frac{\partial}{\partial n'}\frac{1}{|\mathbf{r} - \mathbf{r}'|} \tag{47}$$

and

$$\phi_m(\mathbf{r}) = \frac{1}{4\pi\epsilon}\int_V \frac{(\mathbf{r} - \mathbf{r}')\cdot d\mathbf{m}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\,. \tag{48}$$

This result, with the neglect of all time dependence, has been obtained by Barr et al. (2). In the following section we generalize the result to include time-dependent terms, and in addition we extend equation 46 to apply to an arbitrary number of regions.

C. *Multiple Regions*

Consider now volumes V_i ($i = 1 \cdots I$) separated by surfaces S_{ij} ($j = 1 \cdots J(i)$) in the notation of section II C. Furthermore we label the potential in V_i at the surface S_{ij} as ϕ_{ij}, and, as before, the normal derivative in V_i at the surface S_{ij} in the direction from (i) to (j) is written ∂_{ij}. Note that $\phi_{ij} = \phi_{ji}$ but $\partial_{ij}\phi \neq -\partial_{ji}\phi$. Recall that the left-hand side of equation 54 is $-4\pi\phi(\mathbf{r})$, $-2\pi\phi(\mathbf{r})$, or 0 according as $\mathbf{r}$ is within the Green's theorem volume, on the surface S or outside the surface. Let $\mathbf{r}$ lie on S_{ij}. First we choose the Green's theorem volume V to be V_i. Then a minor generalization of equation 40 is

$$2\pi\phi_{ij}(\mathbf{r}) = -\int_{V_i}\frac{1}{r}\nabla'^2\phi(\mathbf{r}')\, dV' + \sum_{k=1}^{J(i)}\int_{S_{ik}}\left[-\phi_{ik}(\mathbf{r}')\partial'_{ik}\left(\frac{1}{r}\right) + \left(\frac{1}{r}\right)\partial'_{ik}\phi(\mathbf{r}')\right] dS'_{ik}\,. \tag{49}$$

Next let V be V_j, and, with $\mathbf{r}$ still on S_{ij}, we obtain

$$2\pi\phi_{ji}(\mathbf{r}) = -\int_{V_j} \frac{1}{r} \nabla'^2\phi(\mathbf{r}')\, dV' + \sum_{k=1}^{J(j)} \int_{S_{jk}} \left[-\phi_{jk}(\mathbf{r}')\partial'_{jk}\left(\frac{1}{r}\right) + \left(\frac{1}{r}\right) \partial'_{jk}\phi(\mathbf{r}') \right] dS'_{jk}, \quad (50)$$

and finally for $V = V_\iota$, $\iota \neq i, j$, and with $\mathbf{r}\epsilon S_{ij}$, we get

$$0 = -\int_{V_\iota} \frac{1}{r} \nabla'^2\phi(\mathbf{r}')\, dV' + \sum_{k=1}^{J(\iota)} \int_{S_{\iota k}} \left[-\phi_{\iota k}(\mathbf{r}')\partial'_{\iota k}\left(\frac{1}{r}\right) + \left(\frac{1}{r}\right) \partial'_{\iota k}\phi(\mathbf{r}') \right] dS'_{\iota k}. \quad (51)$$

(In general, there are many equations like equation 51.) The first integrals in equations 49–51 are the source terms which will be written

$$-\epsilon_i \int_{V_i} \frac{dV'}{r} \nabla'^2\phi(\mathbf{r}') = 4\pi\epsilon_i\phi_{m_i}(\mathbf{r}). \quad (52)$$

If each surface S separates two volumes, each surface appears twice in the three equations 49–51. The boundary conditions

$$\epsilon_i\partial_{ij}\phi + \epsilon_j\partial_{ji}\phi = \omega_{ij} \quad (18\ a)$$

and

$$\sigma_i\partial_{ij}\phi + \sigma_j\partial_{ji}\phi = -\frac{\partial\omega_{ij}}{\partial t} \quad (19\ a)$$

can be used to eliminate the potential gradients from equations 49–51. Multiply equation 49 by ϵ_i, equation 50 by ϵ_j, and each of equations 51 by ϵ_ι and add:

$$2\pi(\epsilon_i + \epsilon_j)\phi_{ij}(\mathbf{r}) = 4\pi \sum_{\iota} \epsilon_\iota\phi_{m_\iota}(\mathbf{r}) + \sum_{(k\iota)} \int_{S_{k\iota}} \left[-\phi_{k\iota}\{\epsilon_k\partial'_{k\iota} + \epsilon_\iota\partial'_{\iota k}\} \frac{1}{r} + \frac{1}{r} \{\epsilon_k\partial'_{k\iota} + \epsilon_\iota\partial_{\iota k}'\}\phi(\mathbf{r}') \right] dS'_{k\iota} \quad (53)$$

$$= 4\pi \sum_{\iota} \epsilon_\iota\phi_{m_\iota}(\mathbf{r}) + \sum_{(k\iota)} \int_{S_{k\iota}} \left[-\phi_{k\iota}\{\epsilon_k\partial'_{k\iota} + \epsilon_\iota\partial'_{\iota k}\} \frac{1}{r} + \frac{1}{r}\, \omega_{k\iota} \right] dS'_{k\iota} \quad (54)$$

using equation 18 *a*. The summation (ι) is over all volumes V_ι, and the summation ($k\iota$) is over all surfaces. The field point $\mathbf{r}$ is on S_{ij}. Similarly, multiplying equations

49–51 by the appropriate conductivity and adding, we obtain:

$$2\pi(\sigma_i + \sigma_j)\phi_{ij}(\mathbf{r}) = 4\pi \sum_{\iota} \sigma_\iota \phi_{m_\iota}(\mathbf{r}) + \sum_{(k\iota)} \int_{S_{k\iota}} \left[-\phi_{k\iota}\{\sigma_k \partial'_{k\iota} + \sigma_\iota \partial'_{\iota k}\} \frac{1}{r} + \frac{1}{r}\left(\frac{-\partial \omega_{k\iota}}{\partial t}\right)\right] dS'_{k\iota} \quad (55)$$

where equation 19 *a* has been used. All references to the surface charge densities $\omega_{k\iota}$ can be eliminated by adding equation 55 and the time derivative of equation 54 to get

$$(\sigma_i + \sigma_j)\phi_{ij}(\mathbf{r}) + (\epsilon_i + \epsilon_j)\frac{\partial}{\partial t}\phi_{ij}(\mathbf{r}) = 2\sum_{\iota}\left\{\sigma_\iota \phi_{m_\iota}(\mathbf{r}) + \epsilon_\iota \frac{\partial}{\partial t}\phi_{m_\iota}(\mathbf{r})\right\} + \sum_{(k\iota)} \int_{S_{k\iota}} \left\{(\sigma_k - \sigma_\iota)\phi_{k\iota}(\mathbf{r}') + (\epsilon_k - \epsilon_\iota)\frac{\partial}{\partial t}\phi_{k\iota}(\mathbf{r}')\right\} K'_{k\iota}(\mathbf{r}, \mathbf{r}')\, dS'_{k\iota} \quad (56)$$

with

$$K'_{k\iota}(\mathbf{r}, \mathbf{r}') = -\frac{1}{2\pi}\partial'_{k\iota}\frac{1}{|\mathbf{r} - \mathbf{r}'|}, \quad (57)$$

just the adjoint of the kernel for the charge integral equation defined in equation 28. If the time dependence is neglected, equation 56 is an integral equation for the potential on all the boundaries *S*. (Again the relative size of the time-dependent term on the right depends on the factor $(\epsilon_k - \epsilon_\iota)/(\sigma_k - \sigma_\iota)$, and the importance of these terms is hard to estimate.)

Note that for an infinite homogeneous medium $\sigma_i = \sigma_j$ and $\epsilon_i = \epsilon_j$ everywhere, and equation 56 is just the potential due to the sources.

If the torso is modeled as a set of arbitrarily shaped regions, each region internally homogeneous but differing from others in its conductivity and dielectric properties, then equation 56 is an exact relation satisfied by the potentials on the surfaces. Barr et al. (2) have obtained and solved numerically a two-region equation neglecting time dependence. For static problems, equation 56 with the time dependence omitted is the exact, continuous form of Barr's discrete equation.

V. SUMMARY

The integro-differential equations governing the potential produced on the torso surface by a dipole source were derived from Maxwell's equations. The torso was taken to be arbitrary in shape and to consist of various different, but internally homogeneous, regions. Two forms of the integro-differential equations were obtained: the "charge equation" (equation 35) and the "potential equation" (equation 56). It was not assumed a priori that the effects of the dielectric properties of the medium were negligible. However, after using a variable transformation (equation

34) it was shown that (apart from a scale factor on the source term) the dielectric properties occur in the charge equation in time-dependent terms which are probably negligible for the electrocardiological case. This was also found to be true for the "potential equation." Thus, if the time-dependence is neglected, the problem can be solved without knowing the dielectric properties of the regions.

APPENDIX I

In general **E** may be expressed in terms of a scalar potential ϕ and a vector potential **A**:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}.$$

Here

$$\phi = \frac{1}{4\pi\epsilon_0}\int \frac{[\rho]\,dV}{r}$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{[\mathbf{j}]\,dV}{r}$$

where $[\rho]$ and $[\mathbf{j}]$ are the "retarded" values of the source densities, to allow for the finite propagation speed c of the fields.

Consider the situation where there is a time-varying electric field **E**. For simplicity assume it to be of the form

$$E = \langle E\rangle \exp(i\omega t)$$

where $\langle E\rangle$ is some average value of the amplitude over the region of interest and $\omega = 2\pi f$ where f is a characteristic frequency. (The general case can be constructed by superposition of such Fourier components.)
Then

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0}{4\pi}\int \frac{\partial}{\partial t}(\sigma \mathbf{E})\frac{1}{r}\,dV = \mu_0\sigma\langle E\rangle i\omega \exp(i\omega t)\int\frac{dV}{4\pi r} \sim \omega(\sigma/\epsilon)(\mu_0\epsilon)\langle E\rangle\langle R^2\rangle \exp(i\omega t)$$

where $\langle R\rangle$ is a characteristic dimension for the region of interest, approximately 0.3 m for electrocardiography. But $\mu_0\epsilon = 1/c^2$ and $\tau = \epsilon/\sigma$ so

$$\frac{\partial A}{\partial t} \sim \omega\tau(T/\tau)^2\langle E\rangle \exp(i\omega t)$$

where T is the transit time of a light signal across the region of interest. Using values $f \lesssim 10^3$ cps, $\tau \sim 10^{-8}$ sec, $T \sim 10^{-9}$ sec.,

$$\frac{\partial A}{\partial t} \sim 10^{-6}\langle E\rangle \exp(i\omega t)$$

which is to say that the vector potential contributes only 10^{-6} of the electric field. Thus, we may assume

$$\mathbf{E} = -\nabla\phi$$

so that

$$\nabla \times \mathbf{E} = 0.$$

It may also be noted that retardation effects produce phase shifts of the order of $\omega\tau$, which is $\lesssim 10^{-4}$ in the present case and thus quite negligible.

APPENDIX II

The charge conservation equation

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0$$

using Ohm's Law becomes

$$\sigma(\nabla \cdot \mathbf{E}) + \frac{\partial \rho}{\partial t} = 0$$

and in a dielectric

$$\nabla \cdot \mathbf{E} = \rho/\epsilon$$

so that

$$\rho = \rho_0 \exp(-t/\tau)$$

where

$$\tau = \epsilon/\sigma.$$

For body fat $\epsilon \sim 80\epsilon_0$ and $\sigma \sim 4 \times 10^{-2}$ ohm^{-1} m^{-1} (see references 12–14) so that $\tau \sim 2 \times 10^{-8}$ sec. Furthermore we argue that $\rho = 0$ at all times. Imagine a source turned on at $t = 0$. At $t = 0^+$ the fields are those appropriate to a nonconducting dielectric medium and satisfy Laplace's equation so

$$\nabla^2\phi(t = 0^+) = 0 = \rho(t = 0^+)$$

and the free charge is initially and always 0 within the conducting medium. The current flowing from the sources is divergence-free at all times.

APPENDIX III

It is required to evaluate integrals such as $\frac{\partial}{\partial n} \int_S \psi(\mathbf{r}') \frac{dS'}{|\mathbf{r} - \mathbf{r}'|}$ where S includes the singular point $\mathbf{r}' = \mathbf{r}$. Separate the surface into two parts, a small disk D of radius δ centered at $\mathbf{r}$, and the remainder of S_{12} denoted by $S_{12}(-)$.

$$\frac{\partial}{\partial n}\int \frac{\psi(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dS' = \frac{\partial}{\partial n}\int_{S_{12}(-)} \frac{\psi(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dS' + \frac{\partial}{\partial n}\psi(\mathbf{r})\int_D \frac{dS'}{|\mathbf{r}-\mathbf{r}'|}$$

where ψ has been taken as constant over the small disk D. The first integrand has no singularity, and the order of integration and differentiation may be interchanged. To evaluate the second integral, consider a point a distance x along the axis of the disk

$$\frac{\partial}{\partial n}\int_D \frac{dS'}{|\mathbf{r}-\mathbf{r}'|} = \lim_{x\to 0}\left[\frac{\partial}{\partial x}\int_0^\delta \frac{2\pi y\,dy}{(x^2+y^2)^{1/2}}\right]$$
$$= \lim_{x\to 0}\,[2\pi(x/\sqrt{x^2+\delta^2}-1)]$$
$$= -\,2\pi.$$

The limit $\delta \to 0$ may now be taken without affecting the result. The integral evaluated over the diminished surface S is called the "proper value" of the integral. The result appears as equation 25.

It is easy to show that $\mathbf{r}' = \mathbf{r}$ singularity in the integrand in equation 2 causes no difficulty in the evaluation of $\phi(\mathbf{r})$ for $\mathbf{r}$ belonging to the surface. Consider a small disk D of radius δ centered at $\mathbf{r}$.

$$\int_S \frac{\omega(\mathbf{r}')\,dS'}{|\mathbf{r}-\mathbf{r}'|} = \int_{S_{12}(-)} \frac{\omega(\mathbf{r}')\,dS'}{|\mathbf{r}-\mathbf{r}'|} + \omega(\mathbf{r})\int_D \frac{dS'}{|\mathbf{r}-\mathbf{r}'|}$$

With the notation of Fig. 3

$$\int_D \frac{dS'}{|\mathbf{r}-\mathbf{r}'|} = \lim_{x\to 0}\int_0^\delta 2\pi y(x^2+y^2)^{-1/2}\,dy$$
$$= \lim_{x\to 0} 2\pi[(x^2+y^2)^{1/2}]_0^\delta = 2\pi\delta$$

which can be made arbitrarily small by choice of δ.

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